Math 255A' Lecture 28 Notes

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1 C^*-Algebras and Normal Functional Calculus

1.1 C^*-algebras

Definition 1.1. On an algebra $\mathscr{A}$ over $\mathbb{C}$, an **involution** is a map $\mathscr{A} \to \mathscr{A} : a \mapsto a^*$ such that

1. $(a^*)^* = a$,
2. $(ab)^* = b^* a^*$,
3. $(\lambda a + b))^* = \overline{\lambda} a^* + b^*$ for all $\lambda \in \mathbb{C}$, $a, b \in \mathscr{A}$.

This this $\mathscr{A}$ is a ***-algebra**.

Definition 1.2. A Banach algebra with an involution is a C^***-algebra** if

$$\|a\|^2 = \|a^* a\| \qquad \forall a \in \mathscr{A}.$$

Example 1.1. Operators on a Hilbert space form a C^*-algebra:

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^* T x, x \rangle \leq \|T^* T\| \|x\|^2.$$

Example 1.2. $\mathcal{B}_0(H)$ is a C^*-algebra (without identity, unless $\dim H < \infty$).

Example 1.3. If X is compact and Hausdorff, then $C_{\mathbb{C}}(X)$ is a C^*-algebra with $f^* := \overline{f}$.

Henceforth, we will only deal with unital C^*-algebras.

Proposition 1.1. *Let $\mathscr{A}$ be a C^*-algebra. Then for all $a \in \mathscr{A}$, $\|a^*\| = \|a\|$. If $\mathscr{A}$ is unital, then $1^* = 1$ and $\|1\| = 1$.*

Proof. We have $\|a\|^2 = \|a^*\| \|a\| \leq \|a^*\| \|a\|$, which gives $\|a\| \leq \|a^*\|$. Switching a and a^*, we get the other inequality.

Suppose $a \in \mathscr{A}$. Then $1^* a = (a^* 1)^* = (a^*)^* = a$ (and same for right multiplication), so $1^* = 1$. This gives $\|1\|^2 = \|1^* 1\| = \|1\|$, so $\|1\| = 0$ or 1. But this is a norm, so $\|1\| = 1$. ☐

1.2 Self-adjoint, normal, and unitary elements

Definition 1.3. $a \in \mathscr{A}$ is

- **self-adjoint** if $a = a^*$,
- **normal** if $aa^* = a^*a$
- **unitary** if $a^* = a^{-1}$.

Proposition 1.2. *Let* $a \in \mathscr{A}$.

1. *If* a *is invertible, then* a^* *is invertible, and* $(a^*)^{-1} = (a^{-1})^*$.
2. $a = x + iy$*, where* x, y *are self-adjoint.*
3. *If* u *is unitary,* $\|u\| = 1$.
4. *If* a *is normal, its spectral radius is* $r(a) = \|a\|$.

Proof. 1. We have $a^*(a^{-1})^* = (a^{-1}a)^* = 1^* = 1$.

2. Let $x = \frac{a+a^*}{2}$ and $y = \frac{a-a^*}{2i}$.
3. $\|u\|^2 = \|u^*u\| = 1$.
4. We know that $r(a) = \lim_n \|a^n\|^{1/n}$. In particular, we can take a subsequence with powers of 2. We have

$$\|a^{2^k}\|^{2^{-k}} = \|a^{2^{k-1}}a^{2^{k-1}}\|^{2^{-k}} = \|a^{2^{k-1}}\|^{2^{-(k-1)}} = \cdots = \|a\|.$$

So $\lim_k \|a^{2^k}\|^{2^{-k}} = \|a\|$. □

Proposition 1.3. *Let* $h : \mathscr{A} \to \mathbb{C}$ *be a nonzero homomorphism. Then*

1. *If* $a = a^*$*, then* $h(a) \in \mathbb{R}$*. In particular, if* $\mathscr{A}$ *is abelian,* $\sigma(a) \subseteq \mathbb{R}$.
2. $h(a^*) = \overline{h(a)}$.
3. $h(a^*a) \geq 0$.
4. *If* u *is unitary, then* $|h(u)| = 1$.

Proof. 1. We know $\|h\|_{A^*} \leq 1$. Let $t \in \mathbb{R}$, and consider $h(a + it)$. We have

$$\begin{aligned}|h(a) + it|^2 &= |h(a+it)|^2 \\ &\leq \|a + it\|^2 \\ &= (a+it)^*(a+it)\end{aligned}$$

$$
\begin{aligned}
&= \|(a-it)(a+it) \\
&= \|a^2+t^2\| \\
&\leq \|a^2\| + t^2.
\end{aligned}
$$

If $h(a) = x+iy$, then we get $x^2+(y+t)^2 \leq \|a\|^2 + t^2$ for all t. This gives us $x^2+y^2+2yt \leq \|a\|^2$ for all t. So we get $y = 0$.

2. If $a = a+iy$, where x, y are self-adjoint, then $a^* = x - iy$. Now apply h.

3. $h(a^*a) = h(a^*)h*(a) = |h(a)|^2$.

4. We have $1 = uu^*$. Now apply h. □

1.3 The Gelfand Transform and functional calculus for normal elements

The extra structure here makes it clear why the spectral theorem is true.

Theorem 1.1. *If $\mathscr{A}$ is an abelian C^*-algebra, then the Gelfand transform $\mathscr{A} \to C(\Sigma)$ is an isometric *-isomorphism,*

Proof. It preserves the involution because

$$\widehat{a^*}(h) = h(a^*) = \overline{h(a)} = \overline{\widehat{a}}(h).$$

If $a \in \mathscr{A}$, then a is normal, so $\|\widehat{a}\|_{\sup} = r(a) = \|a\|$; so the transform is isometric.

To check that this is surjective, by the Stone-Weierstrass theorem, we need only check that $\widehat{\mathscr{A}}$ separates points. If $h_1 \neq h_2$, then let $a \in \mathscr{A}$ be such that $h_1(a) \neq h_2(a)$. Then $\widehat{a}(h_1) \neq \widehat{a}(h_2)$. □

This gives us a full functional calculus: if $\mathscr{A}$ is any abelian C^*-subalgebra of $\mathcal{B}(H)$, then there exists an isometric *-algebra isomorphism $C(\Sigma) \to \mathscr{A}$, namely the inverse of the Gelfand transform. If $N \in \mathcal{B}(H)$ is normal, then $C^*(N) := \overline{\{p(N, N^*) : p \in C[z, \overline{z}]\}}$ is an abelian C^*-algebra which contains N. So normal operators are precisely the ones that have a functional calculus like this.

Proposition 1.4. *In this example, $\Sigma_{C^*(N)}$ is homeomorphic to $\sigma(N) \subseteq \mathbb{C}$ under the homeomorphism $\widehat{N} : \Sigma_{C^*(N)} \to \mathbb{C}$.*

Proof. We know that $\widehat{N}(\Sigma) = \{h(N) : h \in \Sigma\} = \sigma(N)$. We need to check that if $\widehat{N}(h_1) = \widehat{N}(h_2)$, then $h_1 = h_2$. We have $h_1(N) = h_2(N)$, so

$$h_1(N^*) = \overline{h_1(N)} = \overline{h_2(N)} h_2(N^*).$$

So h_1, h_2 agree on any polynomial in N, N^*, which means $h_1 = h_2$. □

Let $\Phi : C(\sigma(N)) \to C^*(N)$ be our functional calculus. For any $f \in C(\sigma(N))$ and $x, y \in H$, consider

$$\langle \Phi(f)x, y\rangle = \int_{\sigma(N)} f \, d\mu_{x,y}$$

for some complex-valued Borel measure $\mu_{x,y}$. The right hand side is defined for all bounded Borel functions f on $\sigma(N)$. Use this to define $\Phi(f)$ for some functions. This extends Φ to a functional calculus from all bounded, Borel functions on $\sigma(N)$ to $\mathcal{B}(H)$. To get a spectral measure of N, use $\Phi(\mathbb{1}_A)$ for all Borel $A \subseteq \sigma(N)$.

Remark 1.1. We can look at abelian algebras generated by multiple commuting operators. There is a form of the spectral theorem in that setting, too.